

L^2 -stability of explicit schemes for incompressible Euler equations

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Abstract

We present an original study on the numerical stability of explicit schemes solving the incompressible Euler equations on an open domain with slipping boundary conditions. Relying on the skewness property of the non-linear term, we demonstrate that some explicit schemes are numerically stable for small perturbations under the condition $\delta t \leq C\delta x^{2r/(2r-1)}$ where r is an integer, δt the time step and δx the space step.

1 Introduction

In order to achieve stability of a numerical scheme solving the incompressible Euler equations in a divergence-free discretisation frame, we seek for a criterion that link the time and the space steps. During unsteady incompressible fluid simulations with the help of divergence-free wavelets [2], we observed the CFL-like condition: $\delta t \leq C\delta x^{4/3}$ for a centered scheme order two (4.2), presented in [3, 2]. Either bibliographical data don't give a satisfying explanation to this phenomena [5, 4], either it relies on a Von Neumann stability analysis for the advection equation where velocity is a constant vector [7], which is not our case.

This work takes advantage of ideas presented in R. Temam [5, 4] for its context: incompressible Euler equations with the use of the skewness property (lemma 3.1). But, at the difference with references [5, 4], we are not in the context of finite elements, but assume a divergence-free space of discretisation. Meanwhile, we make the connection with the results of Von Neumann stability for the convection equation [7].

In the following Note, we manage to derive the propagation in L^2 norm of a small perturbation ε_n of a regular solution \mathbf{u}_n of the numerical explicit scheme 2.1 to solve the incompressible Euler equations (1.1) in a relatively accurate way.

In this case, we establish a stability criterion of the form $\delta t \leq C\delta x^{2r/(2r-1)}$ with r an integer and C an explicitly computable constant, for various explicit schemes. Contrarily to Von Neumann stability [7] which assumes infinite or periodic domain and constant advection velocity, we establish this stability condition solely under regularity assumptions on the velocity.

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The Euler equations modelise incompressible fluid flows with no viscous term:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p = 0, \quad \operatorname{div} \mathbf{u} = 0 \quad (1.1)$$

The use of the Leray projector \mathbb{P} which is the L^2 -orthogonal projector on the divergence-free space, allows us to remove the pressure term:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbb{P}[(\mathbf{u} \cdot \nabla) \mathbf{u}] = 0 \quad (1.2)$$

We start from a discretisation \mathbf{u}_n of the solution \mathbf{u} in time and in space. Then we consider a perturbed solution $\mathbf{u}_n + \varepsilon_n$. Assuming the regularity of \mathbf{u} and the consistency of the scheme, we are interested in the evolution of the perturbation ε_n for different explicit schemes in time.

2 Discretisation in time and in space

In order to solve equation (1.2) numerically, we discretise the solution $\mathbf{u}_n = \mathbf{u}(n\delta t)$ in a divergence-free space $V_{\operatorname{div}0}(\delta x)$. Such spaces appear in spectral codes using the Fourier transform [1], or are produced in a stable way thanks to divergence-free wavelets [6, 2]. The parameter δx stands for the smallest space step in the discretisation space $V_{\operatorname{div}0}(\delta x)$. Hence, every function \mathbf{u}_n in $V_{\operatorname{div}0}(\delta x)$ satisfies:

$$\|\partial_i \mathbf{u}_n\|_{L^p} \leq C(p) \delta x^{-1} \|\mathbf{u}_n\|_{L^p}$$

where $1 \leq p \leq +\infty$, and $\partial_i \mathbf{u}_n$ denotes a partial space derivative of \mathbf{u}_n . The constant $C(p)$ can often be taken equal to 1, and we will do so in the following.

Concerning the discretisation in time, we consider different explicit schemes proceeding in several steps, like Runge-Kutta schemes:

$$\mathbf{u}_{n(0)} = \mathbf{u}_n, \quad \mathbf{u}_{n(\ell)} = \sum_{i=0}^{\ell-1} a_{\ell i} \mathbf{u}_{n(i)} - \sum_{i=0}^{\ell-1} b_{\ell i} \delta t \tilde{\mathbb{P}}[(\mathbf{u}_{n(i)} \cdot \nabla) \mathbf{u}_{n(i)}] \quad \text{for } 1 \leq \ell \leq k, \quad \mathbf{u}_{n+1} = \mathbf{u}_{n(k)} \quad (2.1)$$

where $\tilde{\mathbb{P}}$ stands for the orthogonal projector on the discretisation space $V_{\operatorname{div}0}(\delta x)$. One can notice that $\tilde{\mathbb{P}} \circ \mathbb{P} = \mathbb{P} \circ \tilde{\mathbb{P}} = \tilde{\mathbb{P}}$.

3 L^2 -stability condition for a small perturbation

Actually, the small error ε_n that we introduce corresponds to oscillations at the smallest scale in space $V_{\operatorname{div}0}(\delta x)$. This stability error propagates and may increase at each time step. In what follows, we demonstrate that under some precise CFL-like conditions, the L^2 norm of this small error ε_n is amplified such that:

$$\|\varepsilon_{n+1}\|_{L^2} \leq (1 + C\delta t) \|\varepsilon_n\|_{L^2} \quad (3.1)$$

where C is a constant that neither depends on δx nor on δt .

Thus, after a time elapse T , the error increases at most exponentially as a function of the time:

$$\|\varepsilon_{t_0+T}\|_{L^2} \leq (1 + C\delta t)^{T/\delta t} \|\varepsilon_{t_0}\|_{L^2} \leq e^{CT} \|\varepsilon_{t_0}\|_{L^2} \quad (3.2)$$

For the following stability study, we will need the skewness property of the transport term. This property is utilized for the stability of the incompressible Navier-Stokes equations in [4].

Lemma 3.1 Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^d$, $H^1(\Omega)$ denoting the Sobolev space on the open set $\Omega \subset \mathbb{R}^d$, be such that $(\mathbf{u} \cdot \nabla) \mathbf{v}, (\mathbf{u} \cdot \nabla) \mathbf{w} \in L^2$. If $\mathbf{u} \in \mathbf{H}_{\text{div},0}(\Omega) = \{\mathbf{f} \in (L^2(\Omega))^d, \text{div } \mathbf{f} = 0\}$, then

$$\langle \mathbf{v}, (\mathbf{u} \cdot \nabla) \mathbf{w} \rangle_{L^2(\Omega)} = - \langle (\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w} \rangle_{L^2(\Omega)}$$

Corollary 3.1 With the same assumptions as in lemma 3.1,

$$\langle \mathbf{v}, (\mathbf{u} \cdot \nabla) \mathbf{v} \rangle_{L^2(\Omega)} = \int_{\mathbf{x} \in \Omega} \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{v} d\mathbf{x} = 0$$

In the scheme (2.1), we denote by $\varepsilon_{n(\ell)}$ the stability error at level ℓ . Then, under the condition $\delta t = o(\delta x)$ and for ε_n small enough, most of the terms appearing in the expression of $\varepsilon_{n(\ell)}$ become negligible compared with:

- the terms $\delta t^i \tilde{F}^i(\varepsilon_n)$ where $\tilde{F}(\varepsilon_n) = \tilde{\mathbb{P}}[(\mathbf{u}_n \cdot \nabla) \varepsilon_n]$ and $\tilde{F}^i = \tilde{F} \circ \tilde{F} \circ \dots \circ \tilde{F}$, i times.
- the term $\delta t \tilde{\mathbb{P}}[(\varepsilon_n \cdot \nabla) \mathbf{u}_n]$,

Hence, for the scheme (2.1), we find the following expression for $\varepsilon_{n(\ell)}$:

$$\varepsilon_{n(\ell)} = \sum_{i=0}^{\ell} \alpha_{\ell i} \delta t^i \tilde{F}^i(\varepsilon_n) + \beta_{\ell} \delta t \tilde{\mathbb{P}}[(\varepsilon_n \cdot \nabla) \mathbf{u}_n] + o() \quad (3.3)$$

where function $o()$ gathers the negligible terms.

$$\alpha_{\ell i} = \sum_{j=i}^{\ell-1} a_{\ell j} \alpha_{ji} + \sum_{j=i-1}^{\ell-1} b_{\ell j} \alpha_{ji-1} \quad \text{and} \quad \beta_{\ell} = \sum_{j=1}^{\ell-1} a_{\ell j} \beta_j + \sum_{j=0}^{\ell-1} b_{\ell j}$$

As a result, we find the following expression for ε_{n+1} :

$$\varepsilon_{n+1} = \sum_{i=0}^k \alpha_i \delta t^i \tilde{F}^i(\varepsilon_n) - \delta t \tilde{\mathbb{P}}[(\varepsilon_n \cdot \nabla) \mathbf{u}_n] + o() \quad (3.4)$$

Starting from this expression and using the fact that according to lemma 3.1,

$$\langle \tilde{F}^i(\varepsilon_n), \tilde{F}^j(\varepsilon_n) \rangle_{L^2(\Omega)} = \begin{cases} 0 & \text{if } i+j = 2\ell+1 \quad \text{for } \ell \in \mathbb{N} \\ (-1)^{\ell-i} \|\tilde{F}^{\ell}(\varepsilon_n)\|_{L^2}^2 & \text{if } i+j = 2\ell \quad \text{for } \ell \in \mathbb{N} \end{cases} \quad (3.5)$$

we compute the L^2 norm of ε_{n+1} as a function of the L^2 norm of ε_n :

$$\|\varepsilon_{n+1} + \delta t \tilde{\mathbb{P}}[(\varepsilon_n \cdot \nabla) \mathbf{u}_n]\|_{L^2}^2 = \sum_{\ell=0}^k S_{\ell} \delta t^{2\ell} \|\tilde{F}^{\ell}(\varepsilon_n)\|_{L^2}^2 + o() \quad (3.6)$$

with

$$S_{\ell} = \sum_{j=-\min(\ell, k-\ell)}^{\min(\ell, k-\ell)} (-1)^j \alpha_{\ell-j} \alpha_{\ell+j} \quad (3.7)$$

For consistency needs of the numerical scheme, we must have $S_0 = 1$. If, on an other hand we suppose $S_1 = S_2 = \dots = S_{r-1} = 0$ and $S_r > 0$, knowing that in the discretised space $V_{\text{div},0}(\delta x)$,

$$\|\tilde{F}^r(\varepsilon_n)\|_{L^2} \leq \|\mathbf{u}_n\|_{L^{\infty}}^r \frac{\|\varepsilon_n\|_{L^2}}{\delta x^r} \quad \text{and} \quad \|\tilde{\mathbb{P}}[(\varepsilon_n \cdot \nabla) \mathbf{u}_n]\|_{L^2} \leq \|\varepsilon_n\|_{L^2} \|\nabla \mathbf{u}_n\|_{L^{\infty}}$$

we derive:

$$\|\varepsilon_{n+1}\|_{L^2} \leq \left(1 + \left(\|\nabla \mathbf{u}_n\|_{L^\infty} + \frac{\delta t^{2r-1} S_r}{2\delta x^{2r}} \|\mathbf{u}_n\|_{L^\infty}^{2r} + o(\cdot)\right) \delta t\right) \|\varepsilon_n\|_{L^2} \quad (3.8)$$

If, on an other hand we assume the consistency, there exist constants A_0 and A_1 such that $\|\mathbf{u}_n\|_{L^\infty} \leq A_0$ and $\|\nabla \mathbf{u}_n\|_{L^\infty} \leq A_1$ when δx and δt go to 0. Hence, the numerical scheme (2.1) is stable for small perturbations under the condition:

$$\delta t \leq C \delta x^{\frac{2r}{2r-1}} \quad (3.9)$$

That brings the following theorem out:

Theorem 3.1 *An order $2p$ scheme solving the incompressible Euler equations is numerically stable for small perturbations at worst under the CFL-like condition:*

$$\delta t \leq C \delta x^{\frac{2p+2}{2p+1}} \quad (3.10)$$

Proof: For an order $2p$ scheme, we have the following equality, point by point:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \delta t \partial_t \mathbf{u}_n + \frac{\delta t^2}{2} \partial_{tt} \mathbf{u}_n + \dots + \frac{\delta t^{2p}}{(2p)!} \partial_t^{2p} \mathbf{u}_n + o(\delta t^{2p})$$

Considering $\partial_t \mathbf{u}_n = \mathbb{P}[(\mathbf{u}_n \cdot \nabla) \mathbf{u}_n]$ and introducing a small perturbation ε_n , leads to:

$$\varepsilon_{n+1} = \varepsilon_n + \delta t \tilde{F}(\varepsilon_n) + \frac{\delta t^2}{2} \tilde{F} \circ F(\varepsilon_n) + \dots + \frac{\delta t^{2p}}{(2p)!} \tilde{F} \circ F^{2p-1}(\varepsilon_n) + \delta t \tilde{\mathbb{P}}[(\varepsilon_n \cdot \nabla) \mathbf{u}_n] + o(\cdot) \quad (3.11)$$

with $F(\varepsilon) = \mathbb{P}[(\mathbf{u}_n \cdot \nabla) \varepsilon]$ and $o(\cdot)$ gathering the terms that are negligible under the condition $\delta t = o(\delta x)$. Then for $q \in [1, p]$,

$$S_q = \sum_{p=0}^{2q} (-1)^{(q-p)} \frac{1}{p!} \frac{1}{(2q-p)!} = \frac{(-1)^q}{(2q)!} \sum_{p=0}^{2q} C_{2q}^p (-1)^p = 0$$

Which allows us to conclude, as stated at line (3.8).

Remark 3.1 *A Von Neumann stability analysis would have proceeded as follows. We compute the evolution of the Fourier mode $\varphi(n\delta t) = \varphi_n e^{i\zeta \cdot \mathbf{x}}$ with $\zeta \in \mathbb{R}^d$, for the advection equation $\partial_t \varphi = -\mathbf{a} \cdot \nabla \varphi$, with $\mathbf{a} = \mathbf{u}$ a constant velocity. As $\nabla \varphi = i\zeta \cdot \varphi$, for the scheme (2.1), we find $\varphi_{n+1} = \xi \varphi_n$ with, as for computation (3.4),*

$$\xi = \sum_{j=0}^k \alpha_j (-i\mathbf{a} \cdot \zeta)^j \quad \text{and then} \quad |\xi|^2 = \sum_{\ell=0}^k S_\ell \delta t^{2\ell} |\mathbf{a} \cdot \zeta|^{2\ell}$$

The coefficients S_ℓ have the same expression (3.7) as when we used the skewness property. As, on an other hand, we have $|\mathbf{a} \cdot \zeta| \leq \|\mathbf{a}\|/\delta x$ in the discretisation space $V_{\text{div}0}(\delta x)$, we find the same stability criterion if we want to have $|\xi| \leq 1 + C\delta t$.

4 Examples

Let $A_0 = \sup_{t \in [0, T], \mathbf{x} \in \Omega} |\mathbf{u}(t, \mathbf{x})|$ and $A_1 = \sup_{t \in [0, T], \mathbf{x} \in \Omega} |\nabla \mathbf{u}(t, \mathbf{x})|$. We propose to apply our stability analysis to some classical schemes.

The simplest example is the Euler explicit scheme, order one in time:

$$\mathbf{u}_{n+1} = \mathbf{u}_n - \delta t \tilde{\mathbb{P}}[(\mathbf{u}_n \cdot \nabla) \mathbf{u}_n] \quad (4.1)$$

For this, we find:

$$\|\varepsilon_{n+1}\|_{L^2} \leq \left(1 + \left(\frac{A_0^2}{2} \frac{\delta t}{\delta x^2} + A_1\right) \delta t\right) \|\varepsilon_n\|_{L^2}, \quad \text{and the CFL : } \delta t \leq 2C \left(\frac{\delta x}{A_0}\right)^2$$

An improved version of this scheme allows us to construct an order two centered scheme:

$$\begin{cases} \mathbf{u}_{n+1/2} = \mathbf{u}_n - \frac{\delta t}{2} \tilde{\mathbb{P}}[(\mathbf{u}_n \cdot \nabla) \mathbf{u}_n] \\ \mathbf{u}_{n+1} = \mathbf{u}_n - \delta t \tilde{\mathbb{P}}[(\mathbf{u}_{n+1/2} \cdot \nabla) \mathbf{u}_{n+1/2}] \end{cases} \quad (4.2)$$

For this scheme, stability is slightly improved:

$$\|\varepsilon_{n+1}\|_{L^2} \leq \left(1 + \frac{\delta t^4}{8\delta x^4} A_0^4 + \delta t A_1\right) \|\varepsilon_n\|_{L^2} \quad \text{hence the CFL : } \delta t \leq 2C^{1/3} \left(\frac{\delta x}{A_0}\right)^{4/3}$$

For Runge-Kutta scheme of order 4:

$$\begin{cases} \mathbf{u}_{n(1)} = \mathbf{u}_n - \frac{\delta t}{2} \tilde{\mathbb{P}}[(\mathbf{u}_n \cdot \nabla) \mathbf{u}_n] \\ \mathbf{u}_{n(2)} = \mathbf{u}_n - \frac{\delta t}{2} \tilde{\mathbb{P}}[(\mathbf{u}_{n(1)} \cdot \nabla) \mathbf{u}_{n(1)}] \\ \mathbf{u}_{n(3)} = \mathbf{u}_n - \delta t \tilde{\mathbb{P}}[(\mathbf{u}_{n(2)} \cdot \nabla) \mathbf{u}_{n(2)}] \\ \mathbf{u}_{n+1} = \mathbf{u}_n - \frac{\delta t}{6} \tilde{\mathbb{P}}[(\mathbf{u}_n \cdot \nabla) \mathbf{u}_n] - \frac{\delta t}{3} \tilde{\mathbb{P}}[(\mathbf{u}_{n(1)} \cdot \nabla) \mathbf{u}_{n(1)}] - \frac{\delta t}{3} \tilde{\mathbb{P}}[(\mathbf{u}_{n(2)} \cdot \nabla) \mathbf{u}_{n(2)}] - \frac{\delta t}{6} \tilde{\mathbb{P}}[(\mathbf{u}_{n(3)} \cdot \nabla) \mathbf{u}_{n(3)}] \end{cases} \quad (4.3)$$

theorem 3.1 allows us to predict a CFL-like condition $\delta t \leq C\delta x^{6/5}$ at worst. Computations show that actually:

$$S_1 = S_2 = 0 \quad \text{and} \quad S_3 = -\frac{1}{72}, S_4 = \frac{1}{576}$$

Hence our study doesn't fully apply to this case.

The order two Adams-Bashford scheme doesn't remain to the definition (2.1). It goes as follows:

$$\mathbf{u}_{n+1} = \mathbf{u}_n - \frac{3}{2} \delta t \tilde{\mathbb{P}}[(\mathbf{u}_n \cdot \nabla) \mathbf{u}_n] + \frac{1}{2} \delta t \tilde{\mathbb{P}}[(\mathbf{u}_{n-1} \cdot \nabla) \mathbf{u}_{n-1}] \quad (4.4)$$

Nevertheless, it is possible to perform computations similar to those of part 3. First, as it is order two, according to theorem 3.1, it is stable at worst under the condition $\delta t \leq C\delta x^{4/3}$. Further computations show that:

$$\|\varepsilon_{n+1}\|_{L^2} \leq \left(1 + \frac{\delta t^4}{4\delta x^4} A_0^4 + \delta t A_1\right) \|\varepsilon_n\|_{L^2} \quad \text{inducing the CFL: } \delta t \leq 2^{2/3} C^{1/3} \left(\frac{\delta x}{A_0}\right)^{4/3}$$

As a conclusion, many usual schemes for simulating the fluid flows verify a stability condition of the type $\delta t \leq C\delta x^{2r/(2r-1)}$ with r an integer.

One can remark that if we have relation (3.11) at order m with no term of the type $\tilde{F} \circ \dots \circ \tilde{F}(\varepsilon_n)$ in $o()$ (which is, for instance, what Runge-Kutta schemes satisfy), then the related scheme will have to verify a CFL-like condition of the type $\delta t \leq C\delta x^{(m+1)/m}$ if $m \equiv 1[4]$ and $\delta t \leq C\delta x^{(m+2)/(m+1)}$ if $m \equiv 2[4]$.

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References

- [1] C. Canuto, M.T. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral methods in fluid dynamics*, Springer-Verlag, New-York, 1988.
- [2] E. Deriaz and V. Perrier, *Direct Numerical Simulation of turbulence using divergence-free wavelets*, Preprint IMPAN 684 <http://www.impan.gov.pl/EN/Preprints/index.html>, 2007.
- [3] R. Kupferman and E. Tadmor, *A fast, high resolution, second-order central scheme for incompressible flows*, Proc. Natl. Acad. Sci. USA Vol. 94, pp. 4848-4852, May 1997 Mathematics.
- [4] M. Marion and R. Temam, Handbook of Numerical Analysis, Vol. VI, *Numerical Methods for Fluids* (Part 1), Elsevier Science, 1998.
- [5] R. Temam, *The Navier-Stokes equations*, North-Holland, Amsterdam, 1984.
- [6] K. Urban, *Wavelet Bases in $H(\text{div})$ and $H(\text{curl})$* , Mathematics of Computation **70**(234): 739-766, 2000.
- [7] P. Wesseling, *Principles of Computational Fluid Dynamics*, Berlin et al., Springer-Verlag 2001.